NAME: ______________________

Blacktown Boys' High School

2024 Year 12

Trial Examination

Mathematics Advanced

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided for this paper
- In Questions in Section II, show all relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 3 – 7)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 10 – 35)

- Attempt Questions 11 – 33
- Allow about 2 hours 45 minutes for this section

Assessor: K. Publico

Students are advised that this is a trial examination only and cannot in any way guarantee the content of format of the 2024 Higher School Certificate Examination.

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1–10.

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1 Which of the following is the range of the function $f(x) = x^2 + 3$?

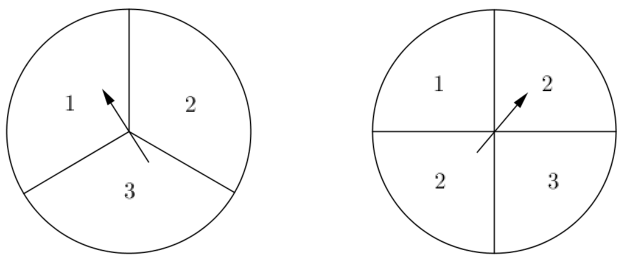
A. $[3, \infty)$

B. $(3, \infty)$

C. $(-\infty, 3)$

D. $(-\infty, \infty)$

2 The two spinners shown are used in a game.



Spinner A *Spinner B*

Each arrow is spun once. The score is the total of the two numbers shown by the arrows. A table is drawn up to show all scores below is partially completed.

	Spinner B				
Spinner A		**1**	**2**	**2**	**3**
	1	2	3		
	2	3			
	3				

What is the probability of getting a score of 4 or more?

A. $\frac{1}{3}$

B. $\frac{1}{2}$

C. $\frac{2}{3}$

D. $\frac{3}{4}$

3 What is the derivative of $\cos(\ln x)$, where $x > 0$?

A. $-\sin\left(\frac{1}{x}\right)$

B. $-\sin(\ln x)$

C. $-\sin\left(\frac{\ln x}{x}\right)$

D. $-\frac{\sin(\ln x)}{x}$

4 The weights of packets of lollies are normally distributed with a mean of 180g. If 99.85% of these packets of lollies have a weight more than 170g, then the standard deviation of the distribution to one decimal place is

A. 3.0

B. 3.1

C. 3.2

D. 3.3

5 Let $a = e^x$. Which expression is equal to $\log_e(a^7)$?

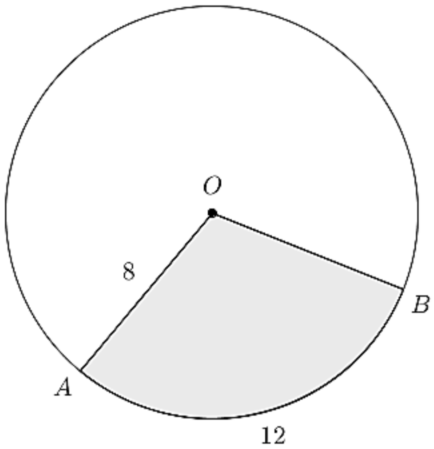
A. x^7

B. $7x$

C. e^{x^7}

D. e^{7x}

6 The circle centred at O has radius 8. Arc AB has length 12 as shown in the diagram.



What is the area of the shaded sector OAB?

A. 48

B. 48π

C. $\frac{64}{3}$

D. $\frac{64}{3}\pi$

7 For which values of x is the curve $f(x) = 2x^3 + x^2$ concave down?

A. $x < -\frac{1}{6}$

B. $x > -\frac{1}{6}$

C. $x < 6$

D. $x > 6$

8 The probability density function $f(x)$ is given by

$$f(x) = \begin{cases} \frac{1}{12}(8x - x^3) & 0 \le x \le 2 \\ 0 & elsewhere \end{cases}$$

The median m of this function satisfies the equation:

A. $-m^4 + 4m^2 - 6 = 0$

B. $m^4 - 16m^2 = 0$

C. $m^4 - 16m^2 + 24 = 0.5$

D. $m^4 - 16m^2 + 24 = 0$

9 Which of the following is equivalent to

$$\frac{\tan^2 x - 1}{\tan^2 x + 1}$$

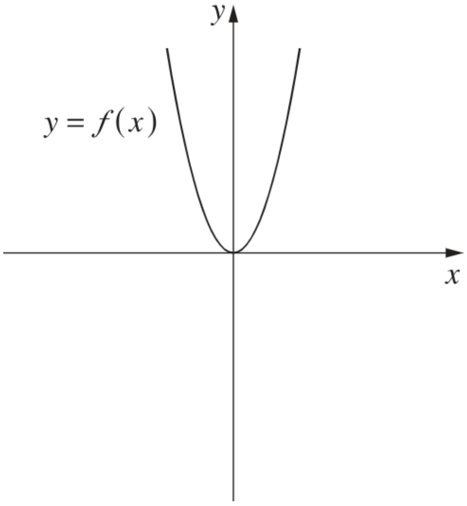
A. $1 - 2\cos^2 x$

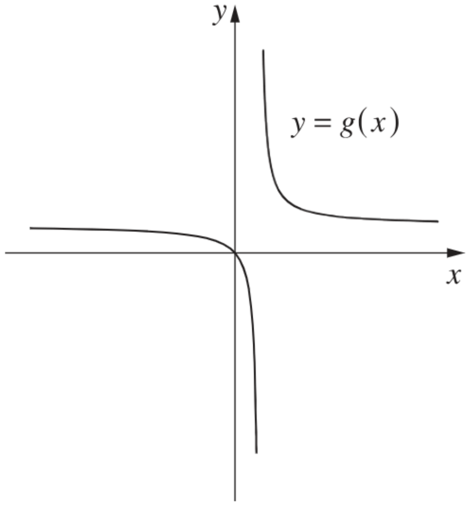
B. $1 - \sec^2 x$

C. $\dfrac{1}{\sin^2 x - \cos^2 x}$

D. $\sec^2 x - \operatorname{cosec}^2 x$

10 The graphs of $y = f(x)$ and $y = g(x)$ are shown.





Which graph best represents $y = f\big(g(x)\big)$?

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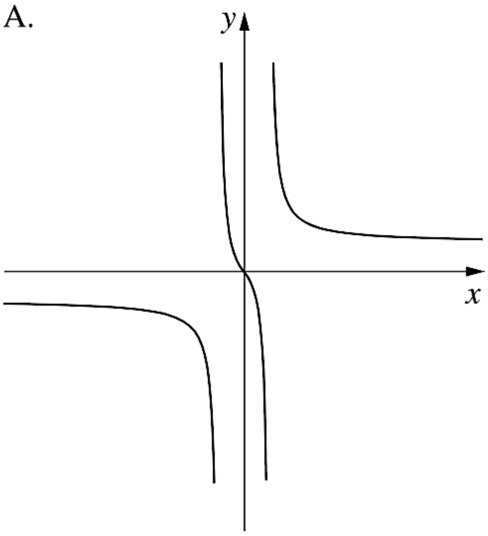
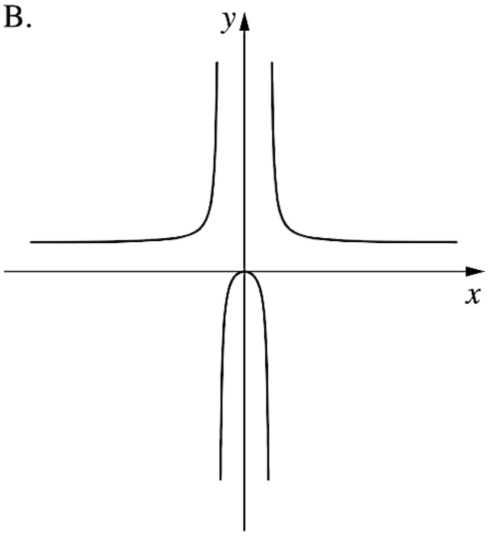
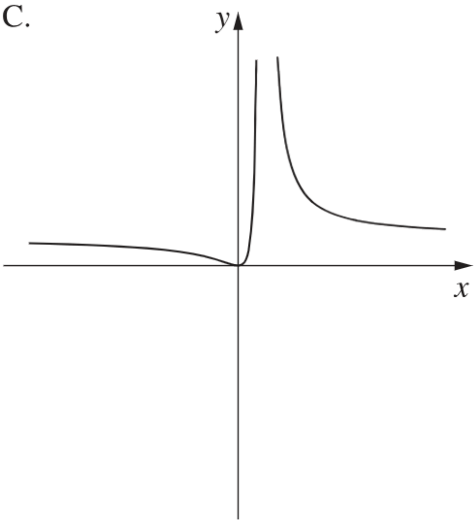


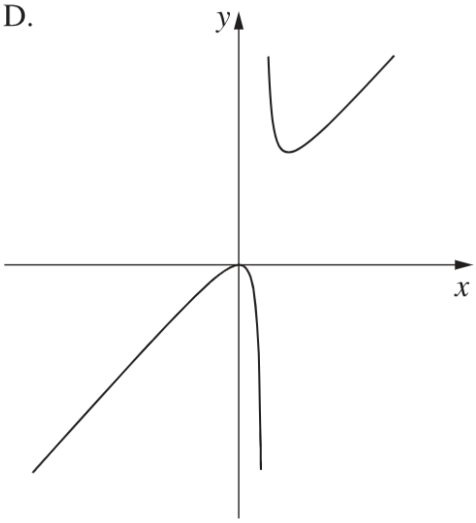
B.



C.



D.



End of Section I

Question 11 (2 marks)

Show that $\cot\dfrac{5\pi}{3} = -\dfrac{1}{\sqrt{3}}$ and hence evaluate $\cos\dfrac{7\pi}{4} \times \cot\dfrac{5\pi}{3}$. **2**

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Question 12 (2 marks)

Calculate the sum of the arithmetic series $17 + 25 + 33 + \cdots + 3089$. **2**

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Question 13 (3 marks)

The table shows the types of study techniques students have said were the most effective for preparing for examinations.

Study Technique	*Frequency*	*Cumulative Frequency*	*Cumulative Percentage*
Practice questions	108	108	36
Flashcards	87	195	65
Mind mapping	45	240	**A**
Note taking	21	261	87
Group work	18	**B**	93
Watch videos	15	294	98
Read textbooks	6	300	100
Total	300		

(a) What are the values of **A** and **B**? **2**

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(b) Determine which are the most significant study techniques by applying the Pareto principle. **1**

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Question 14 (6 marks)

The discrete random variable has probability distribution shown below.

X	0	1	2	3	4	5
$P(X = x)$	0.07	k	0.1	$3k$	$2k$	0.05

(a) Show that the value of k is 0.13. **1**

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(b) Calculate the expected value and variance of X. **2**

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(c) Find $P(X \geq 4)$. **1**

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(d) Find $P(X < 4 | X \geq 2)$. **2**

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Question 15 (3 marks)

The second term of a geometric series is 91. The sixth term is $\dfrac{91}{4096}$. **3**

Find the possible value(s) of the common ratio and the corresponding first term(s).

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Question 16 (2 marks)

Find the equation of the tangent to the curve $y = (4x+2)^3$ at the point $(1, 0)$. **2**

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Question 17 (3 marks)

(a) Differentiate $e^{3x}(1+3x)$. **2**

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(b) Hence, find $\int (2+3x)e^{3x}dx$ **1**

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Question 18 (3 marks)

(a) Use the trapezoidal rule with five function values to estimate **2**

$$\int_{-2}^{2}\sqrt{4-x^2}\,dx$$

Leave the answer in exact form.

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(b) Is the area found in part (a) an overestimation or underestimation? Explain your answer. **1**

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Question 19 (4 marks)

The table below shows the future values of an annuity of $1 for different rates of interest and for different numbers of compounding periods. The contributions made are at the end of each time period.

Future Value Interest Factors

Time Period	***Interest Rate***				
	1%	2%	3%	4%	5%
1	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0100	2.0200	2.0300	2.0400	2.0500
3	3.0301	3.0604	3.0909	3.1216	3.1525
4	4.0604	4.1216	4.1836	4.2465	4.3101
5	5.1010	5.2040	5.3091	5.4163	5.5256
6	6.1520	6.3081	6.4684	6.6330	6.8019
7	7.2135	7.4343	7.6625	7.8983	8.1420
8	8.2857	8.5830	8.8923	9.2142	9.5491

(a) An annuity account is to be opened in order to save $8500 in 2 years time. Equal instalments are to be made at the end of each quarter for 2 years at an interest rate of 16% p.a. compounding quarterly. Find the amount of each quarterly instalment. **2**

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(b) Find the amount in the annuity account after 1 year. **2**

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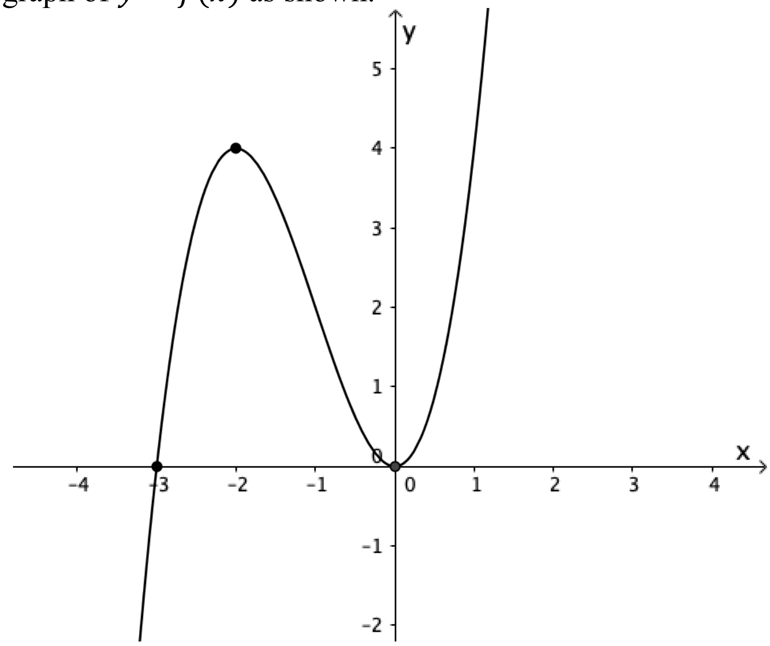
Question 20 (2 marks)

Consider the graph of $y = f(x)$ as shown. **2**



Sketch the graph of $y = \frac{1}{2}f(4x)$ showing the x-intercepts and the coordinates of the turning points.

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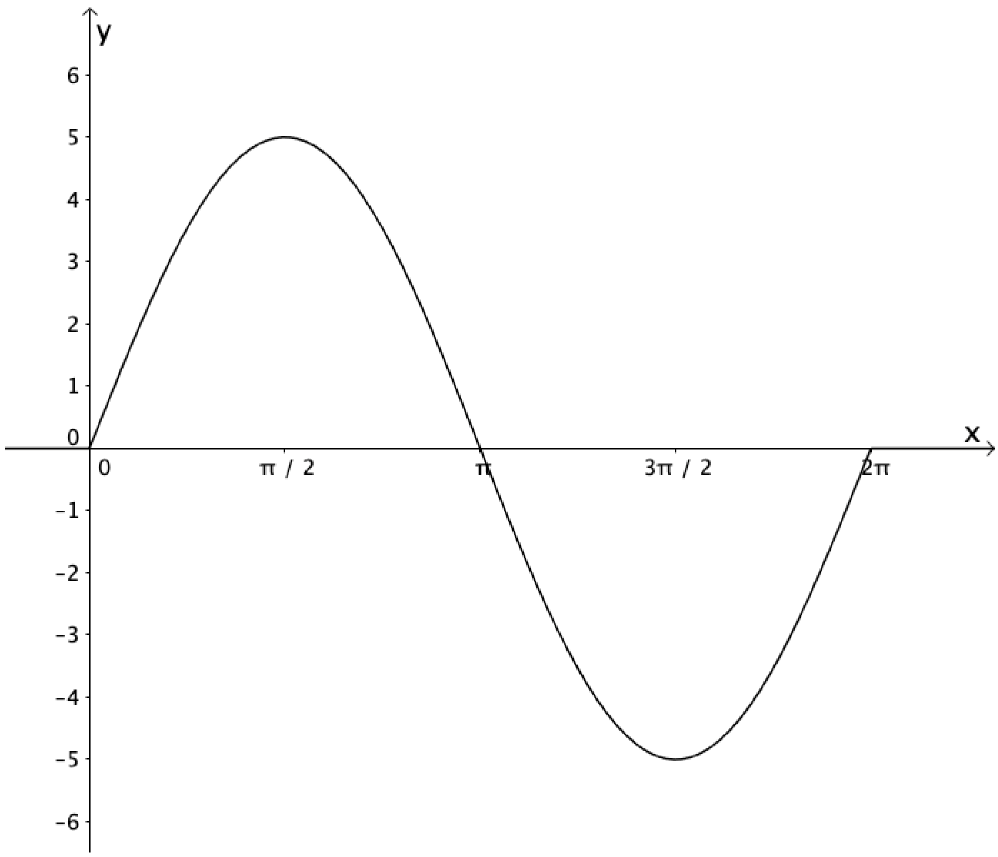
Question 21 (4 marks)

(a) The diagram below shows the graph of $y = 5\sin x$. On the same set of axes sketch $y = 3\cos x$ between $0 \leq x \leq 2\pi$. **1**



(b) Shade the region represented by the integral $\int_{\frac{\pi}{2}}^{\pi} (5\sin x - 3\cos x)dx$ **1**

on the diagram from part (a).

(c) Find the area of the shaded region. **2**

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Question 22 (5 marks)

A farm has a watering system which repeatedly fills a storage tank then empties its' contents to water different sections of the farm. The volume of water V (in cubic metres) in the tank at a time t is given by the equation

$$V = 2 - 2\sqrt{3}\cos t - 2\sin t$$

where t is measured in minutes.

(a) Find $\dfrac{dV}{dt}$. **1**

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(b) Determine whether the tank is initially filling or emptying, showing all calculations. **1**

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(c) At what time does the tank first become completely full and what is its capacity when it is full? **3**

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Question 23 (5 marks)

25 students sit an exam with a total mark out of 40. For each of the 25 students, the number x of the lessons missed and the number y of the marks lost on the test were recorded. The total number of marks lost was 230 more than the total number of lessons missed. The correlation coefficient between lessons missed and marks lost was $r = 0.81$ and the least squares regression line of marks lost on lessons missed has equation $y = 1.5x + 6$.

(a) Describe the nature and strength of the correlation between lessons missed and marks lost. **1**

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(b) Given that the least squares regression line of marks lost on lessons missed passes through the point $(\bar{x}, \bar{y})$, find the mean number of lessons missed and marks lost. **3**

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(c) Find the equation of the least squares regression line of marks gained on lessons missed. **1**

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Question 24 (6 marks)

Consider the curve given by $y = -x^3 + 12x - 7$, for $-5 \leq x \leq 5$.

(a) Find the coordinates of the stationary points and determine their nature. **3**

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(b) Show that there is a point of inflection at $(0, -7)$. **1**

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(c) Sketch the curve for $-5 \leq x \leq 5$, showing all key features. **2**

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Question 25 (2 marks)

A ball dropped from a height of 16 metres rebounds from the floor, rising to a height of 12 metres. At subsequent rebounds the ball rises to a height equal to three-quarters of the previous height. **2**

Find the total number of metres travelled by the ball before it comes to rest.

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Question 26 (3 marks)

A gardener plants apple trees in rows, starting with a row of 14 trees. Each successive row has 6 more trees.

(a) Calculate the number of trees in the 5^{th} row. **1**

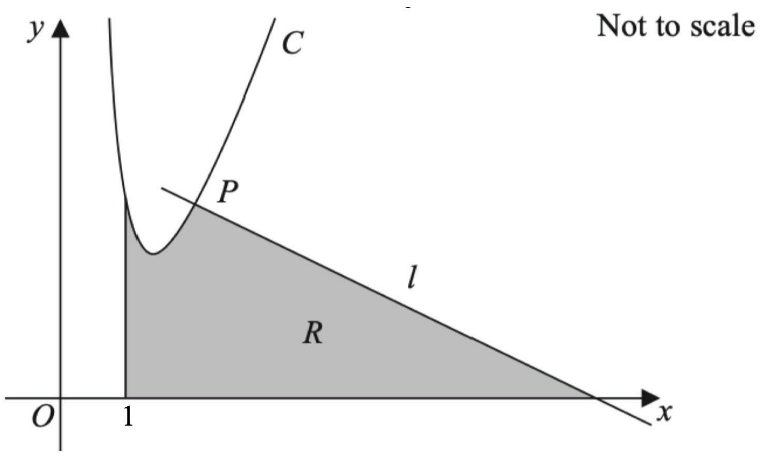
(b) The gardener plants 6020 trees. How many rows are there? **2**

Question 27 (4 marks)

The diagram below shows a sketch of the curve C with equation **4**

$$y = \frac{16}{x^2} + 3x - 4, \qquad x \geq 0$$

The point P lies on the curve and has x-coordinate 4. The line l is the normal to the curve at P.



Find the exact area of the shaded region R.

Question 28 (4 marks)

In a population of giraffes, the heights of adult females and the heights of adult males are each normally distributed. **4**

Information relating to two males from the population is given in Table 1.

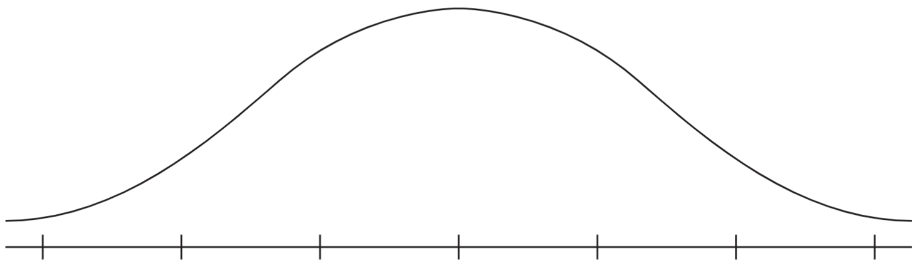
Table 1	*Height (metres)*	*Gender*	*Percentage of males shorter than this giraffe*
	5.3	Male	84%
	4.85	Male	2.5%

The means and standard deviations of adult females and male giraffe heights, in centimetres, are given in table 2.

Table 2		*Mean*	*Standard Deviation*
	Male	m	s
	Female	$0.88m$	$0.9s$

A selected female giraffe is shorter than 97.5% of the population of adult female giraffes.

By first labelling the normal distribution curve below with the heights of the two males given in Table 1, calculate the height of the selected female, in metres, correct to two decimal places.



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Question 29 (3 marks)

The continuous random variable, X, has a probability density function given by

$$f(x) = \begin{cases} \frac{1}{4}\cos\left(\frac{x}{2}\right) & 3\pi \leq x \leq 5\pi \\ 0 & elsewhere \end{cases}$$

(a) Find the mode of X. **1**

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(b) Find the value of a such that $P(X < a) = \dfrac{\sqrt{3}+2}{4}$. **2**

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Question 30 (7 marks)

A particle is moving in a straight line, starting from the origin. At time t seconds, the particle has displacement x metres from the origin and velocity v m/s. The displacement is given by $x = 5\log_e(2t+1) - 3t$.

(a) Find an expression for the velocity v. **1**

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(b) When does the particle come to rest? **2**

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(c) Find the initial acceleration. **2**

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(d) Find the distance travelled by the particle in the first 3 seconds. Answer correct to four decimal places. **2**

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Question 31 (4 marks)

5 people want to adopt an animal at the local shelter. All 5 people have the same probability of adopting a dog $P(D)$ and the same probability of adopting a cat $P(C)$.

It is given that $P(D) = \frac{9}{14}, P(D|C) = \frac{6}{13},$ and $P(C|D) = \frac{1}{3}$.

Dev is one of the five people looking to adopt a pet.

(a) Show that the probability of Dev adopting a cat is $\frac{13}{28}$. **2**

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(b) What is the probability that at least one of the 5 people will NOT adopt a dog. Give your answer correct to two decimal places **2**

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Question 32 (5 marks)

Calvin borrows \$470 000 to purchase a property at 7.1% p.a. where interest is compounded monthly. He makes repayments of \$3300 per month. Interest is calculated just before each payment.

(a) Find the amount owing after the first 2 years. **2**

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Question 32 continued

(b) Find the number of months required to repay the loan. **3**

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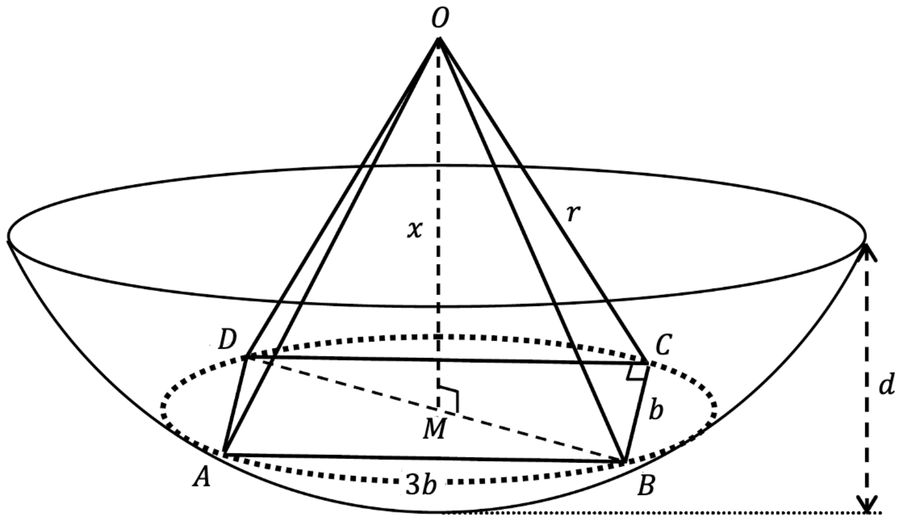
Question 33 (8 marks)

A rectangular pyramid is placed in a bowl of depth d which is part of a sphere with centre O and radius r. The apex of the pyramid coincides with O. The vertices of the base, $ABCD$, lie on the bowl's inner surface as shown in the diagram. The perpendicular from O meets the diagonal DB in its midpoint M.
Let $OM = x$, $CB = b$, and $AB = 3b$.



(a) Show that $DB = 2\sqrt{r^2 - x^2}$ **1**

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(b) Show that the volume of the pyramid is $V = \frac{2}{5}x(r^2 - x^2)$. **2**

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Question 33 continued

(c) Find the value of x that will maximise the volume of the pyramid. **3**

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(d) If the depth of the bowl is $\frac{r}{3}$ units deep, explain why the maximum volume from part (c) cannot be achieved and hence find the greatest volume now possible. **2**

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End of paper

1. Which of the following is the range of the function $f(x) = x^2 + 3$?

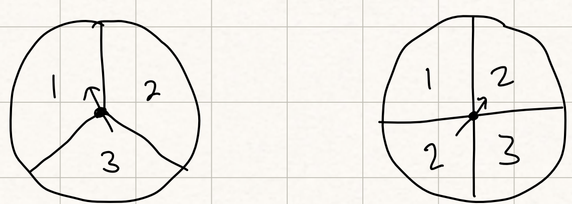
A) $[3, \infty)$ ✓

B) $(3, \infty)$

C) $(-\infty, 3]$

D) $(-\infty, \infty)$

2. The two spinners shown are used in a game



Spinner A Spinner B

Each arrow is spun once. The score is the total of the two numbers shown by the arrows. A table of scores below is partially completed.

		Spinner B			
		1	2	2	3
	1	2	3		
Spinner A	2	3			
	3				

		Spinner B			
		1	2	2	3
	1	2	3	3	4
Spinner A	2	3	4	4	5
	3	4	5	5	6

$\frac{8}{12} = \frac{2}{3}$

What is the probability of getting a score of 4 or more?

A) $\frac{1}{3}$

B) $\frac{1}{2}$

C) $\frac{2}{3}$ ✓

D) $\frac{3}{4}$

(3) What is the derivative of $\cos(\ln x)$, where $x > 0$?

A) $-\sin\left(\frac{1}{x}\right)$

B) $-\sin(\ln x)$

C) $-\sin\left(\frac{\ln x}{x}\right)$

D) $-\frac{\sin(\ln x)}{x}$ ✓

$y = \cos(\ln x)$

$\frac{dy}{dx} = -\sin(\ln x) \times \frac{d}{dx}(\ln x)$

$= -\sin(\ln x) \times \frac{1}{x}$

$= -\frac{\sin(\ln x)}{x}$

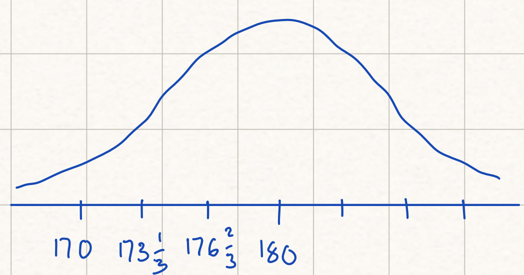
(4) The weights of packets of lollies are normally distributed with a mean of 180 g. If 99.85% of these packets of lollies have a weight more than 170 g, then the standard deviation of the distribution to one decimal place is

A) 3.0 g

B) 3.1 g

C) 3.2 g

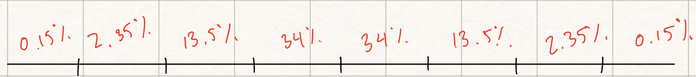
D) 3.3 g ✓



$\therefore$ S.D. $= 3\frac{1}{3}$

$= 3.333\ldots$

$= 3.3$



(5) Let $a = e^x$. Which expression is equal to $\log_e(a^7)$?

A) x^7

B) $7x$ ✓

C) e^{x^7}

D) e^{7x}

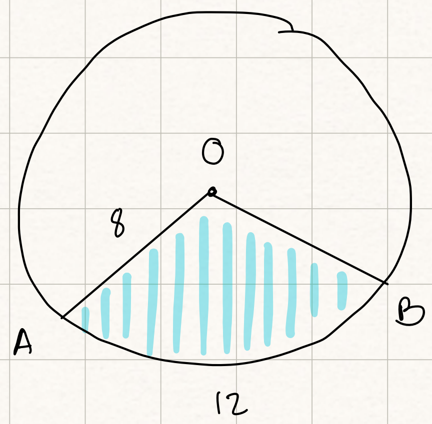
$\log_e(a^4) = \log_e(e^x)^7$

$= \log_e e^{7x}$

$= 7x \log_e e$

$= 7x$

⑥ The circle centred at O has radius 8. Arc AB has length 12 as shown in the diagram.



What is the area of the shaded sector OAB?

A) 48

B) 48π

C) $64/3$

D) $64/3\pi$

$$\text{Arc} = \frac{\theta}{2\pi} \times 2\pi r = r\theta$$

$$12 = 8\theta$$

$$\theta = \frac{12}{8}$$

$$= \frac{3}{2}$$

$$\text{Shaded sector} = \frac{1}{2}r^2\theta$$

$$= \frac{1}{2}(8)^2\left(\frac{3}{2}\right)$$

$$= 48 \quad u^2$$

⑦ For which values of x is the curve $f(x) = 2x^3 + x^2$ concave down?

A) $x < -\frac{1}{6}$ ✓

B) $x > -\frac{1}{6}$

C) $x < -6$

D) $x > 6$

concave down $f''(x) < 0$

$$f'(x) = 6x^2 + 2x$$

$$f''(x) = 12x + 2$$

$$12x + 2 < 0$$

$$12x < -2$$

$$x < -\frac{1}{6}$$

⑧ The probability density function $f(x)$ is given by

$$f(x) = \begin{cases} \frac{1}{12}(8x - x^3) & 0 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

The median m of this function satisfies the equation

A) $-m^4 + 4m^2 - 6 = 0$

B) $m^4 - 16m^2 = 0$

C) $m^4 - 16m^2 + 24 = 0.5$

D) $m^4 - 16m^2 + 24 = 0$ ✓

$$\int_0^m \frac{1}{12}(8x - x^3)\,dx = 0.5$$

$$\frac{1}{12}\left[4x^2 - \frac{x^4}{4}\right]_0^m = 0.5$$

$$\left[4x^2 - \frac{x^4}{4}\right]_0^m = 6$$

$$\left(4m^2 - \frac{m^4}{4}\right) - 0 = 6$$

$$16m^2 - m^4 = 24$$

$$m^4 - 16m^2 + 24 = 0$$

⑨ Which of the following is equivalent to $\dfrac{\tan^2 x - 1}{\tan^2 x + 1}$?

A) $1 - 2\cos^2 x$ ✓

B) $1 - \sec^2 x$

C) $1/\sin^2 x - \cos^2 x$

D) $\sec^2 x - \operatorname{cosec}^2 x$

$$= \frac{\frac{\sin^2 x}{\cos^2 x} - 1}{\frac{\sin^2 x}{\cos^2 x} + 1}$$

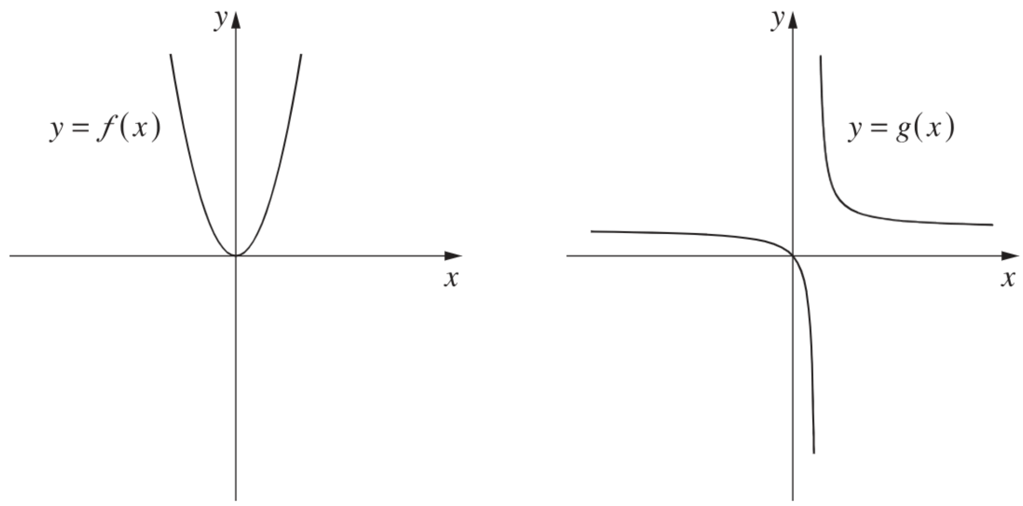
$$= \frac{\sin^2 x - \cos^2 x}{\sin^2 x + \cos^2 x}$$

$$= \sin^2 x - \cos^2 x$$

$$= 1 - \cos^2 x - \cos^2 x$$

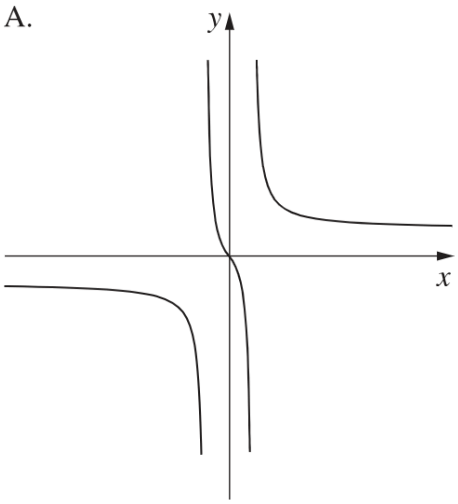
$$= 1 - 2\cos^2 x$$

(10) The graphs of y = f(x) and y = g(x) are shown.

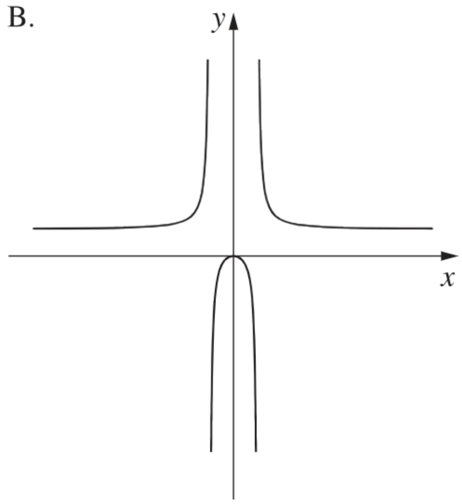


Which graph best represents y = f(g(x)) ?

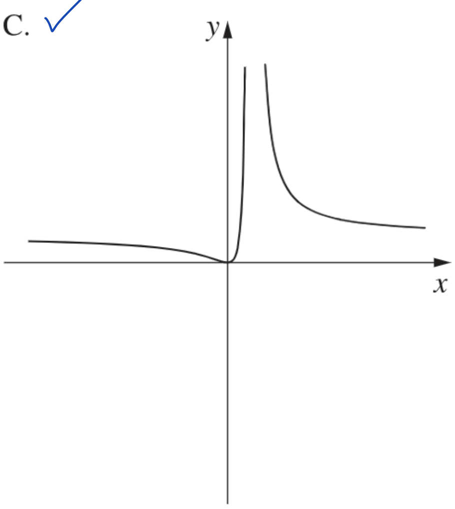
A.



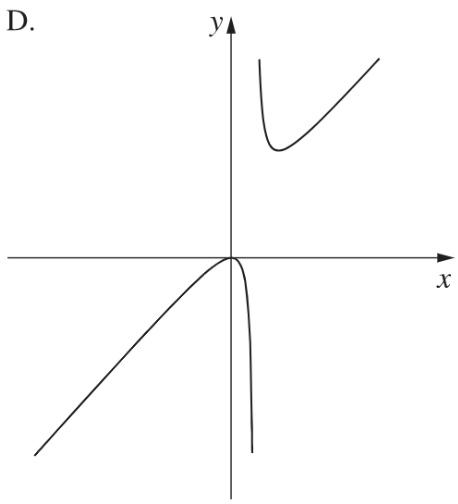
B.



C. ✓



D.



(11) Show that $\cot \frac{5\pi}{3} = -\frac{1}{\sqrt{3}}$ and hence evaluate $\cos \frac{7\pi}{4} \times \cot \frac{5\pi}{3}$ (2)

$$\cot \frac{5\pi}{3} = \frac{1}{\tan \frac{5\pi}{3}}$$

$$= -\frac{1}{\sqrt{3}}$$

(2) correct solution

(1) shows that $\cot \frac{5\pi}{3} = \frac{1}{\tan \frac{5\pi}{3}}$

$$\cos \frac{7\pi}{4} \times \cot \frac{5\pi}{3} = \frac{1}{\sqrt{2}} \times \frac{1}{\tan \frac{5\pi}{3}}$$

$$= \frac{1}{\sqrt{2}} \times -\frac{1}{\sqrt{3}}$$

$$= -\frac{1}{\sqrt{6}}$$

(12) Calculate the sum of the arithmetic series $17 + 25 + 33 \ldots + 3089$. (2)

$a = 17 \quad d = 8 \quad l = 3089$

$$T_n = a + (n-1)d$$

$$3089 = 17 + (n-1)8$$

$$3089 = 17 + 8n - 8$$

$$3089 = 8n + 9$$

$$3080 = 8n$$

$$n = 385$$

$$S_n = \frac{n}{2}(a + l)$$

$$= \frac{385}{2}(17 + 3089)$$

$$= 597905$$

(2) correct solution

(1) finds $n = 385$

(13) The table shows the types of study techniques students have said were the most effective for preparing for examinations.

Study technique	Frequency	Cumulative frequency	Cumulative %
Practice questions	108	108	36
Flashcards	87	195	65
Mind mapping	45	240	80 A
Note taking	21	261	87
Group work	18	279 B	93
Watch videos	15	294	98
Read textbook	6	300	100
Total	300		

a) What are the values of A and B? (2)

A - 80 B - 279

(2) correct solution

(1) one correct value

b) Determine which are the most significant study techniques by applying the Pareto principle. (1)

practice questions, flashcards, mind mapping

(1) correct solution

(14) The discrete random variable has probability distribution

x	0	1	2	3	4	5
$P(X=x)$	0.07	k	0.1	$3k$	$2k$	0.05

a) Show that the value of k is 0.13. (1)

$0.07 + k + 0.1 + 3k + 2k + 0.05 = 1$

$6k = 0.78$

$k = 0.13$

(1) correct solution

b) Calculate the expected value and variance of X. (2)

$E(x) = 0 \times 0.07 + 1 \times 0.13 + 2 \times 0.1 + 3 \times 0.39 + 4 \times 0.26 + 5 \times 0.05$

$= 2.79$

$E(x^2) = 0^2 \times 0.07 + 1^2 \times 0.13 + 2^2 \times 0.1 + 3^2 \times 0.39 + 4^2 \times 0.26 + 5^2 \times 0.05$

$= 9.45$

$Var(x) = E(x^2) - [E(x)]^2$

$= 9.45 - 2.79^2$

$= 1.6659$

(2) correct solution

(1) obtains $E(x) = 2.79$

c) Find $P(x \geq 4)$. (1)

$2(0.13) + 0.05 = 0.31$

(1) correct solution

d) Find $P(x < 4 \mid x \geq 2)$ (2)

$$P(x > 4 \mid x \geq 2) = \frac{P(x < 4 \cap x \geq 2)}{P(x \geq 2)}$$

$$= \frac{P(x=2) + P(x=3)}{1 - P(x=0) - P(x=1)}$$

$$= \frac{0.1 + 0.39}{1 - 0.07 - 0.13}$$

$$= \frac{0.49}{0.8}$$

$$= \frac{49}{80}$$

(2) correct solution

(1) obtains correct numerator or denominator

(15) The second term of a geometric series is 91. The sixth term is $\frac{91}{4096}$.

Find the possible value(s) of the common ratio and the corresponding first term(s). (3)

$T_2 = ar^2 = 91$ (1)

$T_6 = ar^6 = \frac{91}{4096}$ (2)

(2) ÷ (1)

$$\frac{ar^6}{ar^2} = \frac{91/4096}{91}$$

$ar = 91$ (1)

$ar^5 = \frac{91}{4096}$ (2)

(2) ÷ (1)

$$\frac{ar^5}{ar} = \frac{91/4096}{91}$$

$$r^4 = \frac{1}{4096}$$

$$r^2 = \frac{1}{64}$$

$$r = \pm\frac{1}{8}$$

(3) correct solution

(2) solves simultaneously to find r value

(1) forms equations for T_2 and T_6

$a\left(\frac{1}{8}\right) = 91$

$\frac{a}{8} = 91$

$a = 728$

$a\left(-\frac{1}{8}\right) = 91$

$-\frac{a}{8} = 91$

$a = -728$

(16) Find the equation of the tangent to the curve $y = (4x+2)^3$ at the point $(1, 0)$. (2)

$$\frac{dy}{dx} = 3 \times 4 (4x+2)^2$$
$$= 12(4x+2)^2$$

(2) correct solution
(1) correct differentiation

at $(1, 0)$

$$\frac{dy}{dx} = 12(4(1)+2)^2$$
$$= 12(6)^2$$
$$= 432$$

$$m_{tangent} = 432$$
$$y - 0 = 432(x-1)$$
$$y = 432x - 432$$
$$432x - y - 432 = 0$$

(17) a) Differentiate $e^{3x}(1+3x)$. (2)

(2) correct solution
(1) differentiates e^{3x} correctly

$u = e^{3x}$ $\quad$ $v = 1+3x$

$u' = 3e^{3x}$ $\quad$ $v' = 3$

$$\frac{d}{dx} = 3e^{3x} + 3e^{3x}(1+3x)$$
$$= 3e^{3x}(1 + (1+3x))$$
$$= 3e^{3x}(2+3x)$$

b) Hence, find $\int (2+3x)e^{3x}\,dx$ (1)

(1) correct solution

$$\int (2+3x)e^{3x}\,dx = \frac{1}{3}\int 3e^{3x}(2+3x)\,dx$$
$$= \frac{1}{3}e^{3x}(1+3x) + C$$

(18) a) Using the trapezoidal rule, with five function values, to estimate $\int_{-2}^{2} \sqrt{4-x^2}\, dx$ in exact form. (2)

b) Is the area found in part (a) an overestimation or underestimation? Explain your answer. (1)

a)

x	-2	-1	0	1	2
$\sqrt{4-x^2}$	0	$\sqrt{3}$	2	$\sqrt{3}$	0

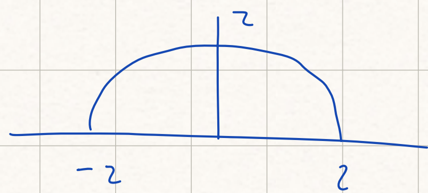
(2) correct solution

$$A = \frac{1}{2}\left(0 + 0 + 2\left(\sqrt{3} + 2 + \sqrt{3}\right)\right)$$
$$= \frac{1}{2}\left(2\left(2\sqrt{3} + 2\right)\right)$$
$$= 2\sqrt{3} + 2$$

(1) applies formula for trapezoidal rule correctly

b) concave down graph

using the rule would give an underestimation



(1) correct solution

(19) a) An annuity account is to be opened in order to save \$8500 in 2 years time. Equal instalments are to be made at the end of each quarter for 2 years at an interest rate of 16% p.a. compounding quarterly. Find the amount of each quarterly instalment. (2)

16% p.a. → 4% per quarter

$2 \times 4 = 8$ quarters

8 quarters at 4% per quarter \$9.2142

Quarterly instalment x is given by $9.2142 \times x = 8500$

$x = 8500 \div 9.2142$

$= \$922.4892\ldots$

$= \$922.49$

(2) correct solution

(1) identifies correct table value.

b) Find the amount in the annuity account after 1 year. (2)

Annuity of \$922.49

4 quarters at 4% per quarter \$4.2465

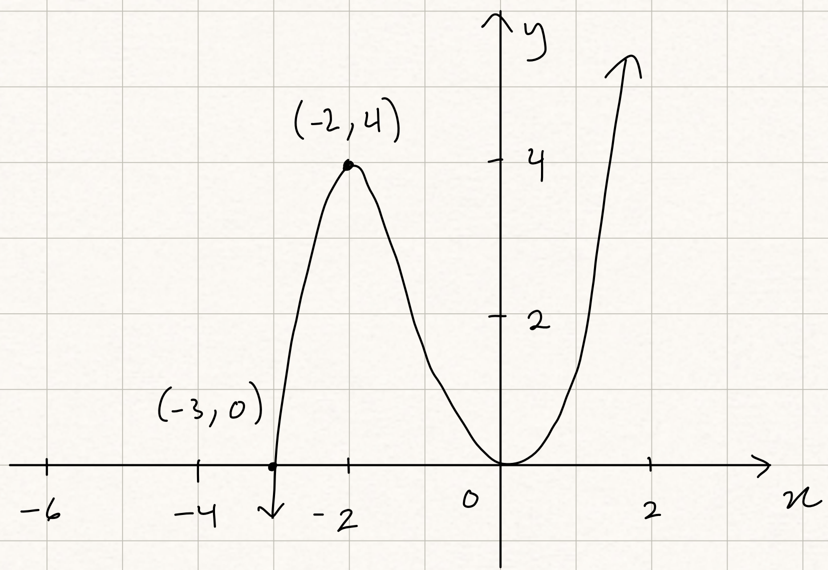
$922.49 \times 4.2465 = \$3917.3537\ldots$

$= \$3917.35$

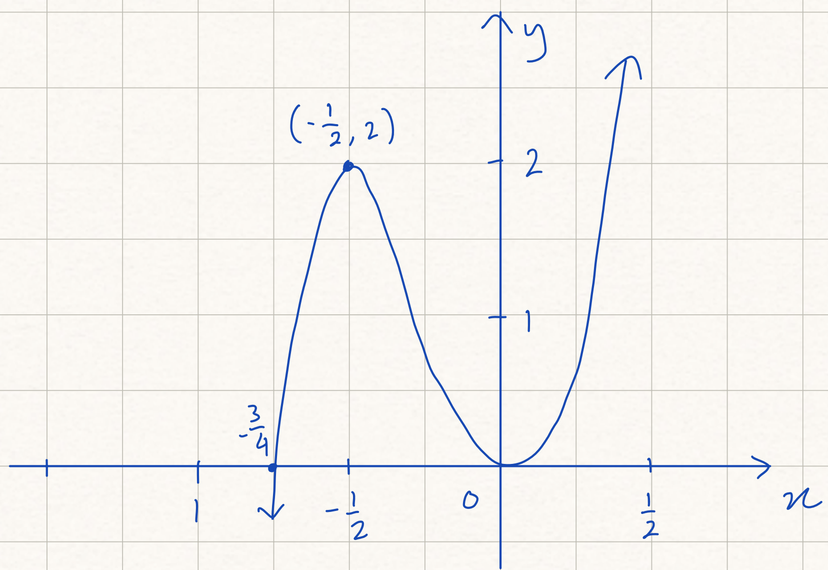
(2) correct solution

(1) identifies correct table value

(20) Consider the graph of $y = f(x)$ as shown.



Sketch the graph of $y = \frac{1}{2} f(4x)$ showing the x-intercepts and the coordinates of the turning points. (2)



$y = \frac{1}{2} f(4x)$

$(-2, 4) \rightarrow (-\frac{1}{2}, 2)$

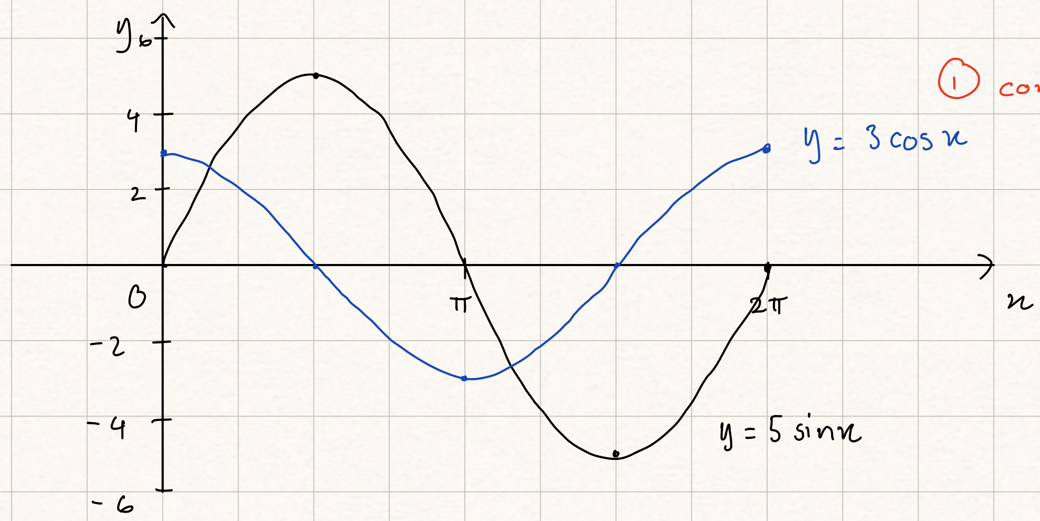
$(-3, 0) \rightarrow (-\frac{3}{4}, 0)$

$(0, 0) \rightarrow (0, 0)$

(2) correct solution

(1) some correct key features shown

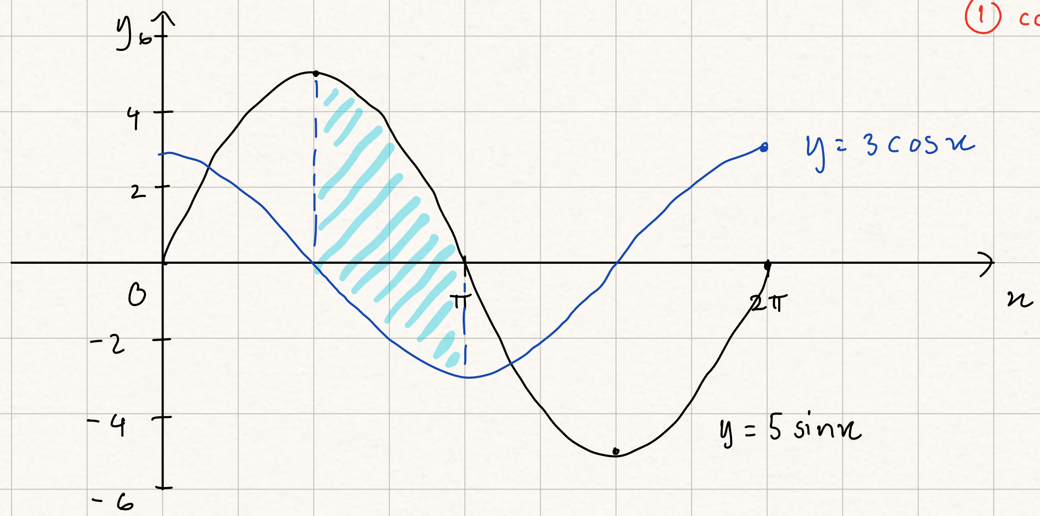
(21) a) The diagram below shows the graphs of $y = 5\sin x$. On the same set of axes sketch $y = 3\cos x$ between $0 \le x \le 2\pi$. (1)



(1) correct solution

b) Shade the region represented by the integral $\int_{\frac{\pi}{2}}^{\pi} (5\sin x - 3\cos x)\,dx$ on the diagram from part (a). (1)

(1) correct solution



c) Find the area of the shaded region. (2)

$$\int_{\frac{\pi}{2}}^{\pi} (5\sin x - 3\cos x)\,dx = \left[-5\cos x - 3\sin x\right]_{\pi/2}^{\pi}$$

$$= \left(-5\cos\pi - 3\sin\pi\right) - \left(-5\cos\tfrac{\pi}{2} - 3\sin\tfrac{\pi}{2}\right)$$

$$= 8 \ u^2$$

(2) correct solution

(1) integrates correctly

(22) A farm has a watering system which repeatedly fills a storage tank then empties it's contents to water different sections of the farm. The volume of water (in cubic metres) in the tank at a time t is given by the equation

$$V = 2 - 2\sqrt{3}\cos t - 2\sin t \quad \text{where } t \text{ is measured in minutes}$$

a) Give an equation for $\frac{dV}{dt}$. (1)

b) Is the tank initially filling or emptying. (1)

c) at what time does the tank first become completely full and what is it's capacity when full? (3)

a) $\frac{dV}{dt} = 2\sqrt{3}\sin t - 2\cos t$

(1) correct solution

b) when $t = 0$

(1) correct solution

$$\frac{dV}{dt} = 2\sqrt{3}\sin(0) - 2\cos(0)$$
$$= 2\sqrt{3}(0) - 2(1)$$
$$= -2$$

$\therefore$ tank is emptying at this time

(3) correct solution

(2) finds $t = \pi/6$ and $t = 7\pi/6$

(1) sets up equation to find $\tan t = \frac{1}{\sqrt{3}}$

c) Full or empty when $\frac{dV}{dt} = 0$

$$2\sqrt{3}\sin t - 2\cos t = 0$$
$$2\sqrt{3}\sin t = 2\cos t$$
$$\frac{\sin t}{\cos t} = \frac{2}{2\sqrt{3}}$$
$$\tan t = \frac{1}{\sqrt{3}}$$
$$t = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6} \ldots$$

$$V = 2 - 2\sqrt{3}\cos t - 2\sin t$$
$$= 2 - 2\sqrt{3}\cos\left(\tfrac{7\pi}{6}\right) - 2\sin\left(\tfrac{7\pi}{6}\right)$$
$$= 6\ m^3$$

as tank is initially emptying, second value corresponds to when full. Tank first full when $t = \frac{7\pi}{6}$ minutes

23 25 students sit an exam with a total mark out of 40. For each of the 25 students, the number x of lessons missed and the number y of marks lost on the test were recorded. The total number of marks lost was 230 more than total number of lessons missed. The correlation coefficient between lessons missed and marks lost was $r = 0.81$ and the least squares regression line of marks lost on lessons missed had equation

$y = 1.5x + 6$.

a) Describe the nature and strength of the correlation between lessons missed and marks lost. (1)

b) Given that the least squares regression line of marks lost on lesson missed passes through the point $(\bar{x}, \bar{y})$, find the mean numbers of lessons missed and marks lost. (3)

c) Find the least squares regression line of marks gained on lessons missed. (1)

a) strong, positive, linear relationship

(1) correct solution

b) $y = 1.5x + 6$

Total marks lost $= 25\bar{y}$

Total lessons missed $= 25\bar{x}$

$25\bar{x} + 230 = 25\bar{y}$ (1)

Sub $(\bar{x}, \bar{y}) \rightarrow \bar{y} = 1.5\bar{x} + 6$ (2)

Sub (2) into (1)

$25\bar{x} + 230 = 25(1.5\bar{x} + 6)$

$25\bar{x} + 230 = 37.5\bar{x} + 150$

$80 = 12.5\bar{x}$

(3) correct solution

(2) finds $\bar{x}$ or $\bar{y}$

(1) obtains $25\bar{x} + 230 = 25\bar{y}$

$\bar{x} = 6.4$

$\bar{y} = 1.5(6.4) + 6$

$\bar{y} = 15.6$

c) Since test is out of 40, Marks gained $= 40 - y$

$\therefore$ least square regression line for marks gained on lessons lost

$$\text{Marks gained} = 40 - (1.5x + 6)$$
$$= 40 - 6 - 1.5x$$
$$= -1.5x + 34$$

1 correct solution

(24) Consider the curve given by $y = -x^3 + 12x - 7$, for $-5 \leq x \leq 5$.

a) Find the coordinates of the stationary points and determine their nature. (3)

$\frac{dy}{dx} = -3x^2 + 12$

$\frac{d^2y}{dx^2} = -6x$

Stationary points at $\frac{dy}{dx} = 0$

(3) correct solution

$0 = -3x^2 + 12$

(2) Finds both stationary points

$3x^2 = 12$

$x^2 = 4$

(1) Differentiates to find $\frac{dy}{dx}$ and obtains

$x = \pm 2$

$x = \pm 2$

at $x = -2$ $\quad y = -(-2)^3 + 12(-2) - 7$

$= -(-8) + (-24) - 7$

$= -23$

at $x = 2$ $\quad y = -(2)^3 + 12(2) - 7$

$= -8 + 24 - 7$

$= 9$

$(-2, -23)$ $\quad \frac{d^2y}{dx^2} = -6(-2)$

$= 12$

$\therefore$ minimum turning point

$(2, 9)$ $\quad \frac{d^2y}{dx^2} = -6(2)$

$= -12$

$\therefore$ maximum turning point.

b) Show that there is a point of inflection at $(0, -7)$. (1)

Inflection point at $\frac{d^2y}{dx^2} = 0$

$0 = -6x$

(1) correct solution

$x = 0$

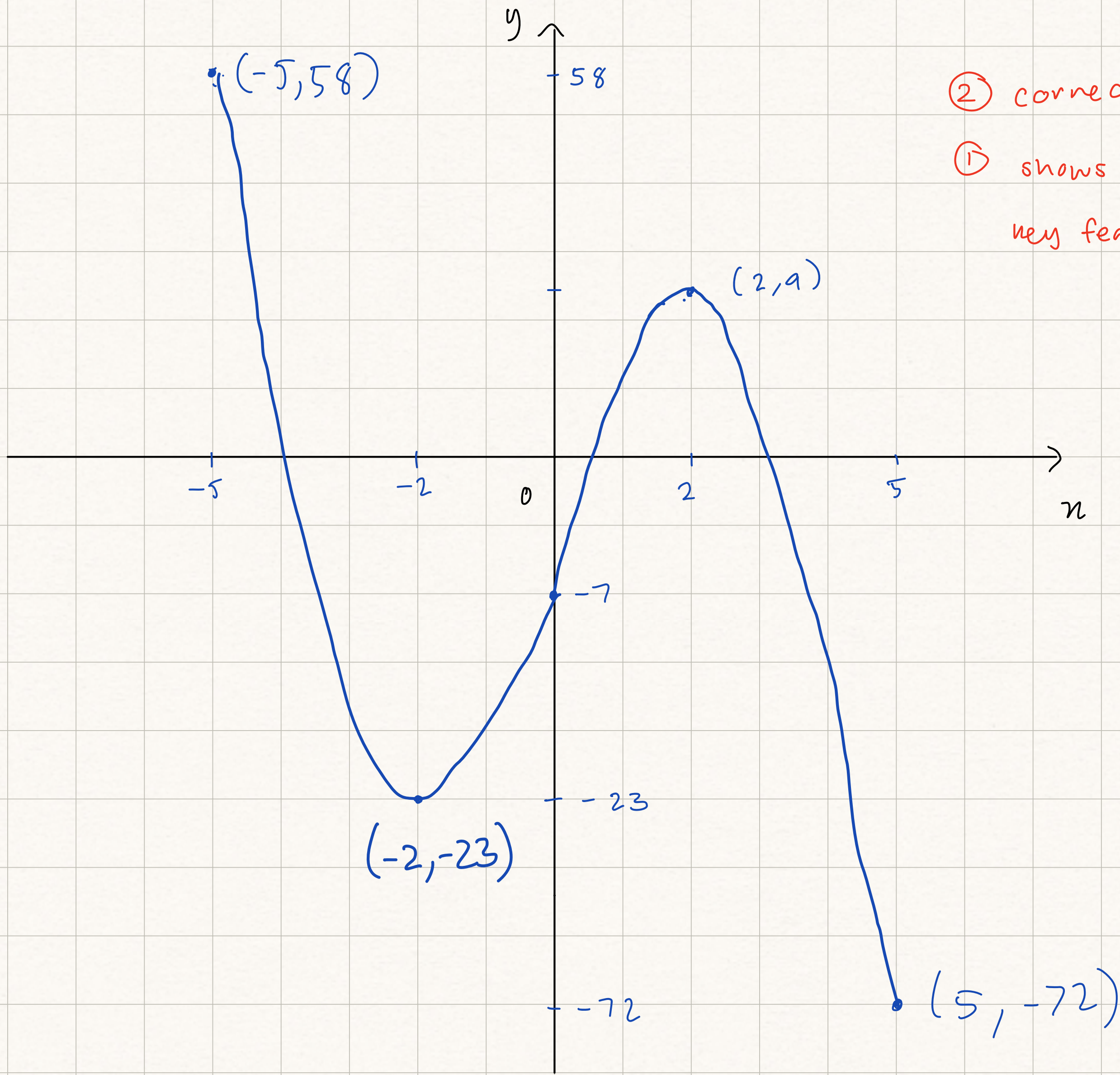
$y = -(0)^3 + 12(0) - 7$

$= -7$

x	-1	0	1
$\frac{d^2y}{dx^2}$	6	0	-6
Concavity	∪	•	∩

$\therefore$ there is a change in concavity at $(0,-7)$.

c) Sketch the curve for $-5 \leq x \leq 5$, showing all key features. (2)



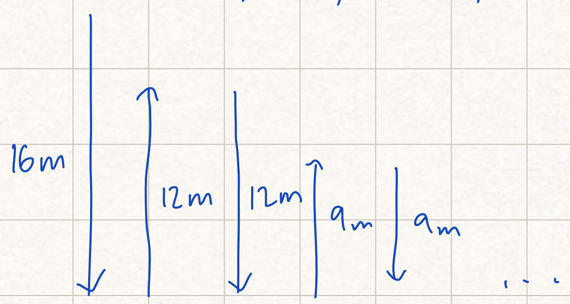
(2) correct solution

(1) shows some key features

(25) A ball dropped from a height of 16 metres rebounds from the floor, rising to a height of 12 metres. At subsequent rebounds the ball rises to a height equal to three-quarters of the previous height.

Find the total number of metres travelled by the ball before it comes to rest. (2)

16, 12, 9, 6.75, ... $a = 12$ $r = \frac{3}{4}$



(2) correct solution

(1) correctly applies limiting sum formula and uses 16

$$\text{Distance travelled} = 16 + 2\left(\frac{12}{1 - \frac{3}{4}}\right)$$

$$= 16 + 96$$

$$= 112 \text{ metres}$$

(26) a gardener plants apple trees in rows, starting with a row of 14 trees. Each successive row has 6 more trees.

a) Calculate the number of trees in the 5th row. (1)

$a = 14 \quad d = 6$

$T_5 = 14 + (5-1)6$

$= 38$ $\therefore$ 38 trees in 5^{th} row

① correct solution

b) The gardener plants 6020 trees. How many rows are there? (2)

$S_n = \frac{n}{2}(2a + (n-1)d)$

② correct solution

① obtains $6n^2 + 22n - 12040 = 0$

$a = 14, \quad d = 6, \quad S_n = 6020$

$6020 = \frac{n}{2}(2(14) + (n-1)6)$

$12\,040 = n(28 + 6n - 6)$

$12\,040 = 28n + 6n^2 - 6n$

$6n^2 + 22n - 12040 = 0$

$n = \dfrac{-22 \pm \sqrt{22^2 - 4(6)(-12040)}}{2(6)}$

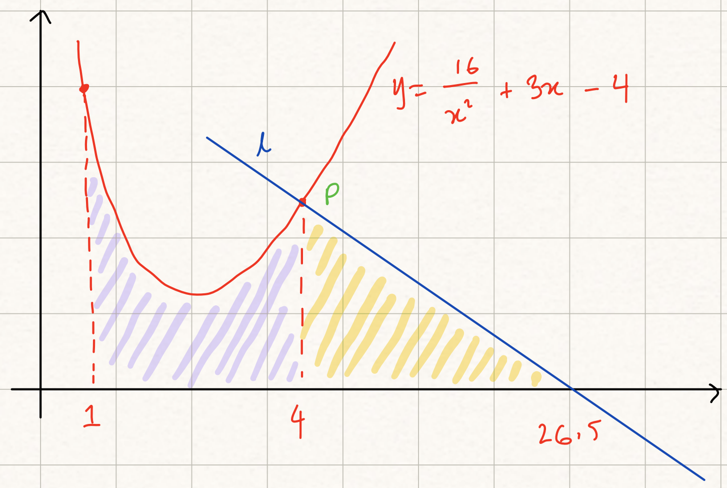
$= \dfrac{-22 \pm \sqrt{289\,444}}{12}$

$= \dfrac{-22 \pm 538}{12}$

$= 43 \quad (n > 0)$

(27) The diagram below shows a sketch of the curve with equation $y = \frac{16}{x^2} + 3x - 4$, $x \geq 0$. The point P lies on the curve and has x-coordinate 4. The line l is the normal to the curve at P.

Find the exact area of the shaded region. (4)



(4) correct solutions

(3) correctly forms integration to find area

(2) finds equation of normal and x-intercept

(1) differentiates to find gradient of normal

$y = \frac{16}{x^2} + 3x - 4$

$y = 16x^{-2} + 3x - 4$

$\frac{dy}{dx} = -32x^{-3} + 3$

at $(4, 9)$ $\quad m_{tangent} = -32(4)^{-3} + 3$

$= 2.5$

$m_{normal} = -\frac{1}{2.5}$

$= -\frac{2}{5}$

Equation of normal

$y - 9 = -\frac{2}{5}(x - 4)$

$5y - 45 = -2x + 8$

$2x + 5y - 53 = 0$

x-intercept $(y = 0)$

$2x + 5(0) - 53 = 0$

$2x = 53$

$x = 26.5$

$$\text{Area} = \int_1^4 \left(\frac{16}{x^2} + 3x - 4\right) dx + \left(\tfrac{1}{2} \times 22.5 \times 9\right)$$

$$= \int_1^4 (16x^{-2} + 3x - 4)\, dx + 101.25$$

$$= \left[\frac{16x^{-1}}{-1} + \frac{3x^2}{2} - 4x\right]_1^4 + 101.25$$

$$= \left[-\frac{16}{x} + \frac{3x^2}{2} - 4x\right]_1^4 + 101.25$$

$$= \left(-\frac{16}{4} + \frac{3(4)^2}{2} - 4(4)\right) - \left(-\frac{16}{1} + \frac{3(1)^2}{2} - 4(1)\right) + 101.25$$

$$= 22.5 + 101.25$$

$$= 123.75$$

(28) In a population of giraffes, the heights of adult females and the heighs of adult males are each normally distributed.

Information relating to two males from the population is given in table 1.

Table 1	Height	Gender	Percentage of males shorter than this giraffe
	5.3 m	Male	84%
	4.85 m	Male	2.5%

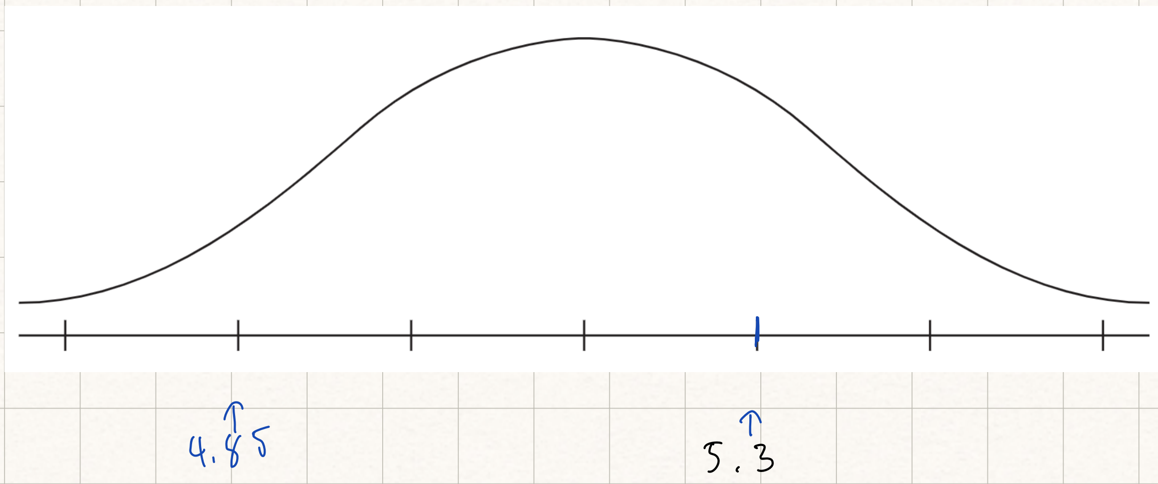
The means and standard deviations of adult female and male giraffe heights, in centimetres, are given in table 2.

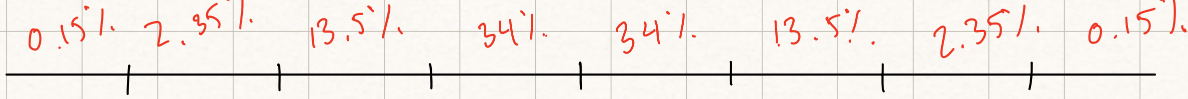
Table 2		Mean	Standard Deviation
	Female	0.88 m	0.95
	Male	m	S

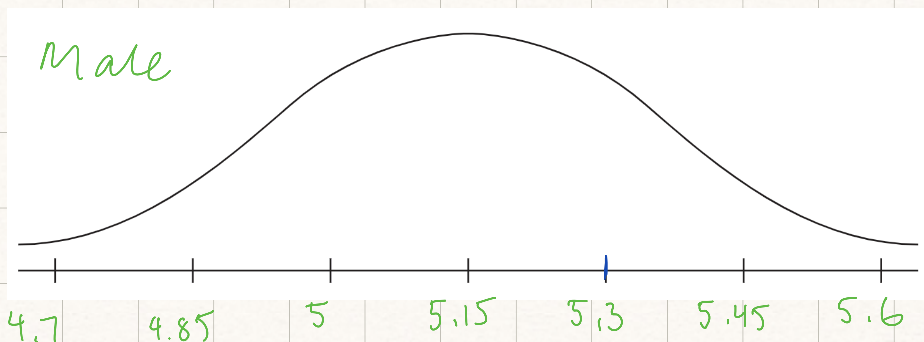
a selected female giraffe is shorter than 97.5% of the population of adult female giraffes.

By first labelling the normal distribution curve below with the heights of the two males given in Table 1, calculate the height of the selected female, in centimetres, correct to 2 decimal places. (4)

Method 1 - Normal Distribution Graphs







Male

mean = 5.15 m

S.D. = 0.15 m

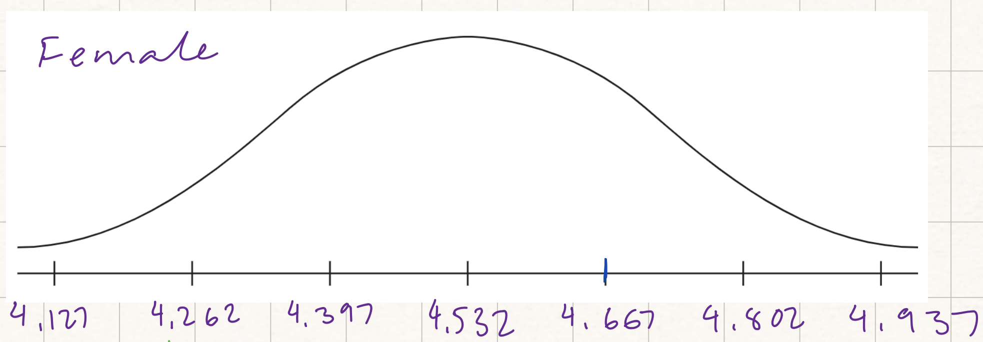
Female

mean = 5.15 × 0.88

= 4.532

S.D. = 0.15 × 0.9

= 0.135



↑

Female shorter than 97.5% of population is 4.262m

= 4.26 m (2 d.p.)

Method 2 - Using z-scores

$$1 = \frac{5.3 - \mu}{\sigma} \qquad -2 = \frac{4.85 - \mu}{\sigma}$$

$$\sigma = 5.3 - \mu \quad (1) \qquad -2\sigma = 4.85 - \mu \quad (2)$$

$(1) - (2)$

$$3\sigma = 0.45$$
$$\sigma = 0.15$$

$$0.15 = 5.3 - \mu$$
$$-5.15 = -\mu$$
$$\mu = 5.15$$

$\therefore$ Male mean $= 5.15$

S.D $= 0.15$

Female

mean $= 5.15 \times 0.88$

$= 4.532$

S.D. $= 0.15 \times 0.9$

$= 0.135$

$$-2 = \frac{x - 4.532}{0.135}$$

$$x - 4.532 = -0.27$$
$$x = 4.262 \text{ m}$$

(4) correct solution

(3) identifies μ and σ of female giraffes

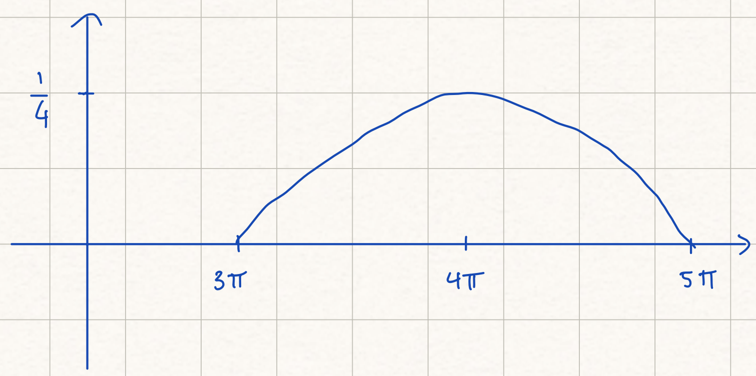
(2) identifies μ and σ of male giraffes

(1) correct labelling of normal distribution curve

(19) The continuous random variable, X, has a probability density function given by

$$f(x) = \begin{cases} \frac{1}{4}\cos\left(\frac{x}{2}\right) & 3\pi \le x \le 5\pi \\ 0 & \text{elsewhere} \end{cases}$$

a) Find the mode of X. (1)



Mode $= 4\pi$

(1) correct solution

b) Find the value of a such that $P(X<a) = \frac{\sqrt{3}+2}{4}$ (2)

$$P(X<a) = \int_{3\pi}^{a} \frac{1}{4}\cos\frac{x}{2}\,dx = \frac{\sqrt{3}+2}{4}$$

(2) correct solution

(1) correct integration

$$\left[\frac{1}{2}\sin\frac{x}{2}\right]_{3\pi}^{a}$$

$$\frac{1}{2}\sin\frac{a}{2} - \frac{1}{2}\sin\frac{3\pi}{2}$$

$$\frac{1}{2}\sin\frac{a}{2} + \frac{1}{2} = \frac{\sqrt{3}+2}{4}$$

$$\frac{1}{2}\sin\frac{a}{2} = \frac{\sqrt{3}}{4} + \frac{2}{4} - \frac{1}{2}$$

$$\frac{1}{2}\sin\frac{a}{2} = \frac{\sqrt{3}}{4}$$

$$\sin\frac{a}{2} = \frac{\sqrt{3}}{2}$$

$$\frac{a}{2} = \frac{7\pi}{3}$$

$$a = \frac{14\pi}{3}$$

30 A particle is moving in a straight line, starting from the origin. At time t seconds, the particle has displacement x metres from the origin and velocity v m/s. The displacement is given by $x = 3t - 5\log_e(2t+1)$

a) Find an expression for the velocity v. (1)

$$\frac{dx}{dt} = v = 5\left(\frac{2}{2t+1}\right) - 3$$

$$= \frac{10}{2t+1} - 3$$

(1) correct solution

b) When does the particle come to rest (2)

at rest when $v = 0$

$$0 = \frac{10}{2t+1} - 3$$

$$\frac{10}{2t+1} = 3$$

$$10 = 3(2t+1)$$

$$10 = 6t + 3$$

$$7 = 6t$$

$$t = \frac{7}{6}$$

(2) correct solution

(1) obtains $10 = 3(2t+1)$

c) Find the initial acceleration (2)

$$v = 10(2t+1)^{-1} - 3$$

$$\frac{dv}{dt} = a = -10 \times 2(2t+1)^{-2}$$

$$= \frac{20}{(2t+1)^2}$$

$$\text{at } t = 0, \quad a = \frac{-20}{(2(0)+1)^2}$$

$$= -20 \text{ m/s}^2$$

(2) correct solution

(1) correct differentiation

d) Find the distance travelled by the particle in the first 3 seconds. answer correct to 4 decimal places. (2)

at $t = 7/6$ $\quad x = 5\ln(2(7/6)+1) - 3(7/6)$

$= 5\ln\left(\frac{10}{3}\right) - 7/2$

$t = 3$ $\quad x = 5\ln(2(3)+1) - 3(3)$

$= 5\ln 7 - 9$

$\therefore$ Total distance travelled $= 2\left(5\ln\left(\frac{10}{3}\right) - 7/2\right) - (5\ln 7 - 9)$

$= 4.310177\ldots$

$= 4.3102$ m

(2) correct solution

(1) finds displacement at $t = 7/6$ or $t = 3$

(31) 5 people want to adopt an animal at the local shelter. all 5 people have the same probability of adopting a dog $P(D)$ and have the same probability of adopting a cat $P(C)$.

It is given that $P(D) = \frac{9}{14}$, $P(D|C) = \frac{6}{13}$, and $P(C|D) = \frac{1}{3}$.

Dev is one of the five people looking to adopt a pet.

a) show that the probability of Dev adopting a cat is $\frac{13}{28}$ (2)

$$P(C|D) = \frac{P(C) \cap P(D)}{P(D)}$$

$$\frac{1}{3} = \frac{P(C) \cap P(D)}{9/14}$$

$$\frac{3}{14} = P(C) \cap P(D)$$

$$P(D|C) = \frac{P(D) \cap P(C)}{P(C)}$$

$$\frac{6}{13} = \frac{3/14}{P(C)}$$

$$P(C) = \frac{3/14}{6/13}$$

$$P(C) = \frac{13}{28}$$

(2) correct solution

(1) obtains $P(C) \cap P(D) = \frac{3}{14}$

b) what is the probability that at least one of the 5 people will NOT adopt a dog. Give your answer to 2 decimal places. (2)

$$P(\text{at least 1 person not adopting a dog}) = 1 - P(\text{all dogs})$$

$$= 1 - \left(\frac{9}{14}\right)^5$$

$$= 0.890207\ldots$$

$$= 0.89$$

(2) correct solution

(1) finds $P(\text{all dogs})$

32) Calvin borrows \$470 000 to purchase a property at 7.1% p.a where interest is compounded monthly. He makes repayments of \$3300 per month. Interest is calculated just before each payment.

i) Find the amount owing after the first 2 years. (2)

ii) Find the number of months required to repay the loan. (3)

i) Repayment = 3300 $\quad r = 7.1\%$ p.a. $\quad n = ?$

$\frac{7.1}{12}\%$ per month

$\frac{71}{12000}$ per month

(2) correct solution

(1) finds expression for A_1 using exact values

$$A_1 = 470\,000\left(1+\frac{71}{12000}\right)^1 - 3300$$

$$= 470\,000\left(\frac{12071}{12000}\right) - 3300$$

$$A_2 = A_1\left(\frac{12071}{12000}\right) - 3300$$

$$= \left[470\,000\left(\frac{12071}{12000}\right) - 3300\right]\left(\frac{12071}{12000}\right) - 3300$$

$$= 470\,000\left(\frac{12071}{12000}\right)^2 - 3300\left(\frac{12071}{12000}\right) - 3300$$

$$= 470\,000\left(\frac{12071}{12000}\right)^2 - 3300\left(\frac{12071}{12000} + 1\right)$$

$$A_3 = A_2\left(\frac{12071}{12000}\right) - 3300$$

$$= \left[470\,000\left(\frac{12071}{12000}\right)^2 - 3300\left(\frac{12071}{12000}+1\right)\right]\left(\frac{12071}{12000}\right) - 3300$$

$$= 470\,000\left(\frac{12071}{12000}\right)^3 - 3300\left(\frac{12071}{12000}^2 + \frac{12071}{12000}\right) - 3300$$

$$= 470\,000\left(\frac{12071}{12000}\right)^3 - 3300\left(\frac{12071}{12000}^2 + \frac{12071}{12000} + 1\right)$$

$\vdots$

$$A_n = 470\,000\left(\frac{12071}{12000}\right)^n - 3300\left(\frac{12071}{12000}^{n-1} + \frac{12071}{12000}^{n-2} + \ldots + \frac{12071}{12000} + 1\right)$$

$$= 470\,000\left(\frac{12071}{12000}\right)^n - 3300\left(1 + \frac{12071}{12000} + \ldots + \frac{12071}{12000}^{n-1}\right)$$

$$= 470\,000\left(\frac{12071}{12000}\right)^n - \frac{3300\left(\left(\frac{12071}{12000}\right)^n - 1\right)}{\frac{12071}{12000} - 1}$$

Hence after 2 years $(n = 24 \text{ months})$

$$A_{24} = 470\,000\left(\frac{12071}{12000}\right)^{24} - \frac{3300\left(\left(\frac{12071}{12000}\right)^{24} - 1\right)}{\frac{12071}{12000} - 1}$$

$= 456654.2456$

$= \$456654.25$ is still owing after 2 years.

ii) Loan repaid = 0 owing.

$A_n = 0$

$$470\,000\left(\frac{12071}{12000}\right)^{n} - \frac{3300\left(\left(\frac{12071}{12000}\right)^{n} - 1\right)}{\frac{12071}{12000} - 1} = 0$$

$$470\,000\left(\frac{12071}{12000}\right)^{n} = \frac{3300\left(\left(\frac{12071}{12000}\right)^{n} - 1\right)}{\frac{12071}{12000} - 1}$$

$$470\,000\left(\frac{12071}{12000}\right)^{n} \times \left(\frac{12071}{12000} - 1\right) = 3300\left(\left(\frac{12071}{12000}\right)^{n} - 1\right)$$

$$\frac{16685}{6}\left(\frac{12071}{12000}\right)^{n} = 3300\left(\frac{12071}{12000}\right)^{n} - 3300$$

$$-\frac{3115}{6}\left(\frac{12071}{12000}\right)^{n} = -3300$$

$$\left(\frac{12071}{12000}\right)^{n} = \frac{3960}{623}$$

$$n = \frac{\ln\left(\frac{3960}{623}\right)}{\ln\left(\frac{12071}{12000}\right)}$$

$= 313.50738\ldots$

$= 314$ months ✓

$= 26$ yrs 2 months

③ correct solution

② collects terms with power n together

① recognises $A_n = 0$

(33) a) Show that $DB = 2\sqrt{r^2 - x^2}$ (1)

$DB = 2MB$

$MB^2 = r^2 - x^2$

$MB = \sqrt{r^2 - x^2}$

$DB = 2\sqrt{r^2 - x^2}$

(1) correct solution

b) Show that the volume of this pyramid is $V = \frac{2}{5}x(r^2 - x^2)$. (2)

$V = \frac{1}{3}Ah$

$= \frac{1}{3}(3b \times b)x$

$= b^2 x$

$DB^2 = b^2 + (3b)^2$

$DB = \sqrt{b^2 + 9b^2}$

$DB = \sqrt{10b^2}$

$\sqrt{10b^2} = 2\sqrt{r^2 - x^2}$

$10b^2 = 4(r^2 - x^2)$

$b^2 = \frac{4(r^2 - x^2)}{10}$

$V = \frac{2}{5}(r^2 - x^2)x$

$V = \frac{2}{5}x(r^2 - x^2)$

(2) correct solution

(1) obtains an expression for DB in terms of b

b) Find the value of x that will maximise the volume of the pyramid. (3)

$V = \frac{2}{5} x r^2 - \frac{2}{5} x^3$

$\frac{dV}{dx} = \frac{2}{5}(r^2 - 3x^2)$

$x = \pm \sqrt{\frac{r^2}{3}}$

$x = \pm \frac{r}{\sqrt{3}}$ $(x > 0)$

Max at $\frac{dV}{dx} = 0$

$0 = \frac{2}{5}(r^2 - 3x^2)$

$3x^2 = r^2$

$x^2 = \frac{r^2}{3}$

(3) correct solution

(2) finds x values

(1) correct differentiation

at $x = r/3$

$\frac{2}{5}\left(r^2 - 3\left(\frac{r}{3}\right)^2\right)$

$\frac{2}{5}\left(r^2 - 3\left(\frac{r^2}{9}\right)\right)$

$\frac{2}{5}\left(r^2 - \frac{r^2}{3}\right)$

$\frac{2}{5} \times \frac{2r^2}{3}$

$\frac{4r^2}{15}$

$x = 2r/3$

$\frac{2}{5}\left(r^2 - 3\left(\frac{2r}{3}\right)^2\right)$

$\frac{2}{5}\left(r^2 - 3\left(\frac{4r^2}{9}\right)\right)$

$\frac{2}{5}\left(r^2 - \frac{4}{3}r^2\right)$

$\frac{2}{5}\left(-\frac{1}{3}r^2\right)$

$-\frac{2r^2}{15}$

x	$\frac{r}{3}$	$\frac{r}{\sqrt{3}}$	$\frac{2r}{3}$
$\frac{dV}{dx}$	$\frac{4r^2}{15}$	0	$-\frac{2r^2}{15}$
Slope	/	—	\

$\therefore$ maximum at $x = \frac{r}{\sqrt{3}}$

c) If the depth d of the bowl is $\frac{r}{3}$ units deep, explain why the max volume from part (b) cannot be achieved and hence find the greatest volume now possible. (2)

From diagram

$$d + x \geq r$$

$$\frac{r}{3} + x \geq r$$

$$x \geq r - \frac{r}{3}$$

$$x \geq \frac{2r}{3} \qquad \therefore x \neq \frac{r}{\sqrt{3}}$$

at $x = \frac{2r}{3}$, $\frac{dV}{dx}$ = negative (decreasing volume)

$\therefore$ max at conditions $x = \frac{2r}{3}$

$$V = \frac{2}{5}\left(\frac{2r}{3}\right)\left(r^2 - \left(\frac{2r}{3}\right)^2\right)$$

$$= \frac{4r}{15}\left(r^2 - \frac{4r^2}{9}\right)$$

$$= \frac{4r}{15}\left(\frac{9r^2}{9} - \frac{4r^2}{9}\right)$$

$$= \frac{4r}{\cancel{15}_3}\left(\frac{\cancel{5}r^2}{9}\right)$$

$$= \frac{4r^3}{27}$$

(2) correct solution

(1) finds $x \geq \frac{2r}{3}$ or states why $\frac{r}{\sqrt{3}}$ would not work